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Clifford algebras obtained by twisting of group algebras

Helena Albuquerque[a, 1], Shahn Majid[b, *]

[a] *Departamento de Matematica-Faculdade de Ciencias e Tecnologia, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal*

[b] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Rd, London, UK E1 4NS*

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Abstract

We investigate the construction and properties of Clifford algebras by a similar manner as our previous construction of the octonions, namely as a twisting of group algebras of $\mathbb{Z}_2^n$ by a cocycle. Our approach is more general than the usual one based on generators and relations. We obtain, in particular, the periodicity properties and a new construction of spinors in terms of left and right multiplication in the Clifford algebra. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1] we have constructed the octonions and other Cayley algebras by a twisting procedure applied to group algebras kG by a 2-cochain F on the group. The failure of the cochain to be a group cocycle controls the nonassociativity of these algebras. On the other hand, in the case when the cochain is actually a group cocycle the associativity will be preserved and we have a conventional 'twisted group ring' [2]. It has already been observed that, in particular, if one uses cocycles which are the quadratic part of the cochains in [1] that define the Cayley algebras then one has in fact their associated Clifford algebras.

* Corresponding author. Royal Society University Research Fellow. www.maths.qmw.ac.uk/~majid
E-mail addresses: lena@mat.uc.pt (H. Albuquerque), s.majid@qmw.ac.uk (S. Majid).
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In this paper, we will explore this construction further, using this point of view to give a new derivation of known results about Clifford algebras and to generalise them. Results include the periodicity theorems for Clifford algebras and a novel construction for their spinor representations. The paper begins in Section 2 by showing how a specific cocycle gives the usual Clifford algebras and many of their properties in a more direct manner. Section 3 contains some constructions for general groups and cocycles that include and generalise the periodicity theorems. Section 4 contains spinor constructions for usual Clifford algebras obtained by our methods. As an example, spinors in 4 dimensions are naturally described as quaternion-valued functions.

1.1. Preliminaries

Let k be a field with characteristic not 2. Let $\mathbf{q}$ be a nondegenerate quadratic form on a vector space V over k of dimension n. It is known [7] that there is an orthogonal basis $\{e_1,\dots,e_n\}$, say, of V such that $\mathbf{q}(e_i)=q_i$ for some $q_i\neq 0$. The Clifford algebra $C(V,\mathbf{q})$, [7], is the associative algebra generated by 1 and $\{e_i\}$ with the relations

$$e_i^2=q_i.1,\qquad e_ie_j+e_je_i=0,\quad \forall i\neq j.$$

We identify k and V inside $C(V,\mathbf{q})$ in the obvious way. The dimension of $C(V,\mathbf{q})$ is 2^n and it has a canonical basis

$$\{e_{i_1}\cdots e_{i_p}\,|\,1\leqslant i_1<i_2\cdots<i_p\leqslant n\}.$$

If we assume that $q_i=\pm 1$ (without loss of generality over $\mathbb{R}$, for example) then for $n=1$ we have two cases: If $q_1=-1$ we have $C(k,q_1)$ the algebraic complex numbers, where we adjoin $i=e_1$ with relation $i^2=-1$. If $q_1=1$ we have the group algebra of $\mathbb{Z}_2$ with e_1 the nontrivial element. Setting $e_\pm=(1\pm e_1)/2$ we have equivalently two projections

$$e_\pm^2=e_\pm,\quad e_+e_-=e_-e_+=0$$

and $C(k,q_1)=k\oplus k$ (the hyperbolic complex numbers over k).

If $n=2$ we have

$$e_1e_2=-e_2e_1,\qquad e_i^2=q_i,\qquad (e_1e_2)^2=-q_1q_2,\qquad (e_1e_2)e_1=-q_1e_2,$$

$$(e_1e_2)e_2=q_2e_1.$$

Hence for $q_1=q_2=-1$ we have the k-algebra of quaternions $\mathbb{H}$ with $i=e_1$, $j=e_2$ and $k=e_1e_2$. If $q_1=q_2=1$ we have the matrix algebra $M_2(k)$ with $e_1=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and $e_2=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. If $q_1=1$ and $q_2=-1$ we have $M_2(k)$ similarly with $e_2=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ instead. Thus,

$$C(0,2)\cong\mathbb{H},\quad C(2,0)\cong C(1,1)\cong M_2(k),$$

where $C(r,s)$ denotes the algebra with r of the $\{q_i\}$ equal to $+1$ and s equal to -1.

2. $C(V)$ as twisting

From [1] we recall that if G is a group and $F: G\times G \to k$ a nowhere-zero function with $F(e,x)=F(x,e)=1$ (a cochain), where e is the group identity, then we define a new algebra $k_F G$ which has the same vector space as the group algebra kG (namely with basis labelled by G) but a different product, namely

$$a\cdot b = F(a,b)ab \quad \forall a,b\in G. \tag{1}$$

This arises naturally as an algebra in the category of comodules of the cotriangular dual-quasiHopf algebra $(kG,\partial F,\mathcal{R})$. When F is a cocycle this gives the category of comodules of a cotriangular usual Hopf algebra $(kG,\mathcal{R})$ built on kG. We first construct this.

Proposition 2.1. *Let $G=\mathbb{Z}_2^n$. There is a 2-cocycle $F\in Z^2(G,k)$ defined by*

$$F(x,y) = (-1)^{\sum_{j<i} x_i y_j}\prod_{i=1}^{n} q_i^{x_i y_i},$$

where $x=(x_1,\ldots,x_n)\in\mathbb{Z}_2^n$ and twists kG into a cotriangular Hopf algebra [10] *with cotriangular structure*

$$\mathcal{R}(x,y) = (-1)^{\rho(x)\rho(y)+x\cdot y},$$

where $\rho(x)=\sum_i x_i\in\mathbb{Z}$ and $x\cdot y$ is the dot product of $\mathbb{Z}_2$-valued vectors.

Proof. We define the cochain F as shown which is manifestly invertible as $q_i\neq 0$ and the identity when either argument is zero. Using the notation $x=(x_1,\ldots,x_n)$, it is clear that

$$\partial F(x,y,z) = \frac{F(x,y)F(x+y,z)}{F(y,z)F(x,y+z)} = \prod_{i=1}^{n} q_i^{x_i y_i + (x_i+y_i)z_i - y_i z_i - x_i(y_i+z_i)} = 1$$

since the (-1) factors certainly do not contribute by linearity. The remaining factors again cancel because the exponents are controlled by a bilinear form. Hence by the twisting theory for Hopf algebras [4,10] the group algebra kG becomes a cotriangular Hopf algebra with the same Hopf algebra structure (namely every element of the group has diagonal coproduct) but with $\mathcal{R}: kG\otimes kG \to k$ defined by $\mathcal{R}(x,y)=F(x,y)/F(y,x)$ on the basis elements of kG (labelled by G). This readily comes out as shown. □

The modification of product (1) from kG to a new algebra $k_F G$ is an application of twisting to comodule algebras [6] and ensures that all its structure maps are morphisms in the symmetric monoidal category [8] of $(kG,\partial F,\mathcal{R})$-comodules. This category is the same as that of G-graded spaces equipped with a generalised transposition given by $\mathcal{R}$ and associativity by ∂F. Since in our case the coboundary $\partial F=1$, the new algebra $k_F G$ remains associative in the usual sense. In this case, $k_F G$ as an algebra is a usual twisted group ring [2] but obtained now with additional structure from our above categorical setting.

Applying this construction, the original product of basis elements in $k\mathbb{Z}_2^n$ corresponds to the addition in $\mathbb{Z}_2^n$. We now define $k_F\mathbb{Z}_2^n$ as the same vector space as $k\mathbb{Z}_2^n$ but with the new product modified by F. For clarity we denote the basis elements of $k\mathbb{Z}_2^n$ by

$$e_x = e_1^{x_1} \cdots e_n^{x_n}, \tag{2}$$

where the unmodified algebra $k\mathbb{Z}_2^n$ is generated by e_i mutually commuting and with $e_i^2 = 1$.

Proposition 2.2. *The algebra $k_F\mathbb{Z}_2^n$ can be identified with $C(V,\mathbf{q})$, i.e. the latter is an algebra in the symmetric monoidal category of $\mathbb{Z}_2^n$-graded spaces defined by $\mathcal{R}$.*

Proof. We identify the elements e_i in the two algebras and note that $e_i \cdot e_j = e_ie_j$ if $i < j$ since $F(x,y) = 1$ in this case (where $x_i = 1 = y_j$ and other entries are zero). Hence we can identify also the basis elements

$$e_1^{x_1} \cdot e_2^{x_2} \cdots e_n^{x_n} = e_x$$

of the two algebras. After that it remains only to check that the products coincide for other products, which can be checked inductively from the generators. Here one has

$$e_i \cdot e_j = -e_ie_j = -e_je_i = -e_j \cdot e_i, \quad e_i \cdot e_i = q_ie_i^2 = q_i1 \quad \forall i, j < i. \qquad \square$$

We can now apply some of the results in [1] albeit in the associative setting since $\phi = \partial F = 1$.

Corollary 2.3. *The algebra $C(V,\mathbf{q})$ is commutative under the generalised transposition or 'braiding' of the symmetric monoidal category defined by $\mathcal{R}$, i.e. $e_x \cdot e_y = \mathcal{R}(x,y)e_y \cdot e_x$. In particular,*

$$e_x \cdot e_y = \begin{cases} e_y \cdot e_x & \text{if } x \cdot y = \rho(x)\rho(y) \pmod 2, \\ -e_y \cdot e_x & \text{else.} \end{cases}$$

Proof. This is an immediate consequence of the above construction in terms of F since $a \cdot b = F(a,b)ab = F(a,b)ba = (F(a,b)/F(b,a))b \cdot a$. In our case it takes the form shown. The result is also clearly true on repeated use of the anticommutation relations of $C(V,\mathbf{q})$ but in our approach these are encoded concisely in $\mathcal{R}$ as stated. Note that either $x \cdot y = \rho(x)\rho(y)$ or $x \cdot y = \rho(x)\rho(y) + 1 \bmod 2$, so two basis elements either commute or anticommute. $\square$

We therefore apply various results about k_FG algebras in [1] to Clifford algebras. For example,

Proposition 2.4. *$\mathcal{R}$ in Proposition* 2.1 *is a group coboundary*, $\mathcal{R}=\partial\theta$ *for cochain*

$$\theta(x)=(-1)^{(1/2)\rho(x)(\rho(x)-1)}$$

with $\theta^2=1$. *Hence by* [1, *Lemma* 3.4] $\Theta(e_x)=\theta(x)e_x$ *is a diagonal anti-involution on the Clifford algebra* $C(V,\mathbf{q})$. *Explicitly*, Θ *is the order-reversal operation*

$$\Theta(e_1^{x_1}\cdots e_n^{x_n})=e_n^{x_n}\cdots e_1^{x_1}.$$

Proof. We show that $\mathcal{R}(x,y)=\theta(x)\theta(y)/\theta(x+y)$, where we write the group structure of $\mathbb{Z}_2^n$ additively. Viewing $\rho(x)=\sum_i x_i$ in $\mathbb{Z}$,

$$\rho(x+y)=\sum_i x_i(1-y_i)+(1-x_i)y_i=\rho(x)+\rho(y)-2x\cdot y \tag{3}$$

and hence

$$\theta(x+y)=(-1)^{(1/2)(\rho(x+y)^2-\rho(x+y))}=(-1)^{(1/2)(\rho(x)^2+\rho(y)^2+2\rho(x)\rho(y)-\rho(x)-\rho(y)+2x\cdot y)}$$
$$=\theta(x)\theta(y)(-1)^{\rho(x)\rho(y)+x\cdot y}.$$

Hence θ defines an anti-involution Θ. In $C(V,\mathbf{q})$ we can identify it with order-reversal on noting that

$$e_n^{x_n}\cdots e_1^{x_1}=e_x(-1)^{\sum_{j<i}x_ix_j}=e_x(-1)^{(1/2)\rho(x)(\rho(x)-1)},$$

where $\rho(x)^2=\sum_{i,j}x_ix_j=2\sum_{j<i}x_ix_j+\sum_i x_i^2=2\sum_{j<i}x_ix_j+\rho(x)$ since $x_i=0,1$. $\square$

It is clear that Θ is an isomorphism between $C(V,\mathbf{q})$ and its opposite algebra, which in turn is of the form $k_{F^{\mathrm{op}}}\mathbb{Z}_2^n$ for $F^{\mathrm{op}}(x,y)=F(y,x)$. We also obtain from ρ the obvious $\mathbb{Z}_2$-grading of Clifford algebras provided by an order 2 automorphism. Here degree zero is the eigenspace with eigenvalue 1 under the involution.

Corollary 2.5. $\sigma(e_x)=(-1)^{\rho(x)}e_x$ *extended linearly is an automorphism of* $C(V,\mathbf{q})$ *in the form above and makes it into a super algebra with* $\rho \bmod 2$ *the super degree. If* n *is even then* σ *is inner*, *being implemented by* $e_{(1,\ldots,1)}=e_1\cdots e_n$.

Proof. The first part is again an immediate consequence of the above construction in terms of F since $k_F\mathbb{Z}_2^n$ is $\mathbb{Z}_2^n$-covariant (it is an algebra in the category of $\mathbb{Z}_2^n$-graded spaces). The map $\rho:\mathbb{Z}_2^n\to\mathbb{Z}$ given by $\rho(x)=\sum_{i=1}^n x_i$ is additive mod 2 (a group homomorphism to $\mathbb{Z}_2$) and therefore induces a functor from the category of $\mathbb{Z}_2^n$-graded spaces to that of $\mathbb{Z}_2$-graded ones. Under this any $\mathbb{Z}_2^n$-graded algebra is also a $\mathbb{Z}_2$-graded or super algebra. The second part is well known from generators and relations. In our case it comes about as

$$e_{(1,\ldots,1)}^{-1}e_xe_{(1,\ldots,1)}=e_{(1,\ldots,1)}^{-1}e_{(1,\ldots,1)}e_x(-1)^{\rho(x)\rho(1,\ldots,1)+x\cdot(1,\ldots,1)}=e_x(-1)^{(n+1)\rho(x)}$$

using the braided-commutativity in Corollary 2.3. Note also that

$$e_{(1,\ldots,1)}^2=F((1,\ldots,1),(1,\ldots,1))=(-1)^{n(n-1)/2}\prod_i q_i. \quad \square \tag{4}$$

In terms of this super-algebra structure one can say that the natural braiding on $C(V,\mathbf{q})$ defined by $\mathcal{R}$ in Proposition 2.1 (with respect to which the Clifford algebra is braided-commutative) is of the form

$$\Psi(e_x \otimes e_y) = \Psi_{\rm super}(e_x \otimes e_y)(-1)^{x\cdot y}, \tag{5}$$

where $\Psi_{\rm super}$ refers to the usual Bose–Fermi statistics or supertransposition. There are of course many other applications of the super-algebra structure.

Corollary 2.6. *$C(V \oplus W, \mathbf{q} \oplus \mathbf{p}) \cong C(V,\mathbf{q})\underline{\otimes} C(W,\mathbf{p})$ as super algebras.*

Proof. This is well known from the point of view of generators and relations. In our description it is clear from the form of F in Proposition 2.1 as

$$F((x,x'),(y,y')) = F(x,y)F(x',y')(-1)^{\rho(x')\rho(y)}$$

where $\{e_x\}$ is a basis of V and $\{e_{x'}\}$ of W (say). Hence the algebra product has the form $(a \otimes c)(b \otimes d) = a \cdot b \otimes c \cdot d(-1)^{\rho(c)\rho(b)}$ for the super tensor product of super algebras. The notation $\mathbf{q} \oplus \mathbf{p}$ indicates zero inner product between V and W. □

Finally, our approach also gives more explicit formulae for the adjoint action and Pin groups. First of all our explicit form of the product means that all the basis elements e_x are invertible in $C(V,\mathbf{q})$, as are generic linear combinations for $q_i = \pm 1$. The latter fact is because the products all have coefficients ± 1 coming from the values of F. Clearly,

$$e_x^{-1} = \frac{e_x}{F(x,x)} = e_x \frac{(-1)^{\sum_{j<i} x_i x_j}}{\prod_{i=1}^n q_i^{x_i}} = \frac{\Theta(e_x)}{\prod_{i=1}^n q_i^{x_i}} \tag{6}$$

in terms of the anti-involution above. We recall that the Clifford group of V consists of the invertible elements of $C(V,\mathbf{q})$ that leave V stable under the adjoint action.

Corollary 2.7. *The adjoint action defined by $\mathrm{ad}_a(e_y) = \sigma(a)e_y a^{-1}$ for all invertible $a \in C(V,\mathbf{q})$ takes the explicit form*

$$\mathrm{ad}_{e_x}(e_y) = (-1)^{\rho(x)(\rho(y)+1)}(-1)^{x\cdot y} e_y.$$

Proof. This follows immediately from the braided-commutativity, i.e. from the form of $\mathcal{R}$ in Proposition 2.1 and Corollary 2.3. Thus, $(-1)^{\rho(x)} e_x \cdot e_y \cdot e_x^{-1} = (-1)^{\rho(x)} (-1)^{\rho(x)\rho(y)+x\cdot y} e_y \cdot e_x \cdot e_x^{-1}$. □

Also, using Θ, one defines $\lambda : C(V,\mathbf{q}) \to k$ by $e_x \sigma \circ \Theta(e_x) = \lambda(e_x)1$. From our formula for inverses we obtain explicitly

$$\lambda(e_x) = (-1)^{\rho(x)} \prod_{i=1}^n q_i^{x_i}. \tag{7}$$

By definition the group $\mathrm{Pin}(V)$ is the subgroup of the Clifford group with $\lambda = \pm 1$ and clearly includes all the e_x when $q_i = \pm 1$. The even part of this is the spin group. These groups map surjectively onto $O(V)$ and $SO(V)$ via ad.

3. Clifford process

We now use the above convenient description of Clifford algebras to express a 'doubling process' similar to the Dickson process for division algebras. Thus, let A be a finite-dimensional algebra with identity 1 and σ an involutive automorphism of A. For any fixed element $q \in k^*$ there is a new algebra of twice the dimension,

$$\bar{A} = A \oplus Av, \quad (a + bv) \cdot (c + dv) = a \cdot c + qb \cdot \sigma(d) + (a \cdot d + b \cdot \sigma(c))v$$

with a new involutive automorphism

$$\bar{\sigma}(a + vb) = \sigma(a) - \sigma(b)v.$$

We will say that $\bar{A}$ is obtained from A by *Clifford process*, see [11]. We consider this initially for not necessarily associative algebras and then find conditions for associativity to be preserved.

Proposition 3.1. *Let G be a finite Abelian group and F a cochain as above, so $k_F G$ is a G-graded quasialgebra. For any $s : G \to k^*$ with $s(e) = 1$ and any $q \in k^*$, define $\bar{G} = G \times \mathbb{Z}_2$ and*

$$\bar{F}(x, yv) = F(x, y) = \bar{F}(x, y), \qquad \bar{F}(xv, y) = s(y)F(x, y),$$

$$\bar{F}(xv, yv) = qs(y)F(x, y), \quad \bar{s}(x) = s(x), \quad \bar{s}(xv) = -s(x)$$

for all $x, y \in G$. Here $x \equiv (x, e)$ and $xv \equiv (x, \eta)$, where η with $\eta^2 = e$ is the generator of the $\mathbb{Z}_2$. If $\sigma(e_x) = s(x)e_x$ is an involutive automorphism then $k_{\bar{F}}\bar{G}$ is the Clifford process applied to $k_F G$.

Proof. We clearly have a new cochain since $\bar{F}(e, xv) = F(e, x) = 1$ and $\bar{F}(xv, e) = s(e)F(x, e) = 1$. The formulae are fixed by reproducing the product of $\bar{A}$ in the involutive case. Thus, $F(x, yv)xyv = x \cdot yv = (x \cdot y)v = F(x, y)xyv$, $F(xv, y)xvy = xv \cdot y = (x \cdot \sigma(y))v = s(y)(x \cdot y)v = s(y)F(x, y)xyv$, etc. $\square$

It is easy to see that the special case where s defines an involutive automorphism on $k_F G$ is precisely the one where $s : G \to k^*$ is a character with $s^2 = 1$.

Proposition 3.2. *For any $s : G \to k^*$ and $q \in k^*$ as above the $k_{\bar{F}}\bar{G}$ given by the generalised Clifford process has associator and braiding*

$$\bar{\phi}(x, yv, z) = \bar{\phi}(x, y, zv) = \bar{\phi}(x, yv, zv) = \phi(x, y, z),$$

$$\bar{\phi}(xv, y, z) = \bar{\phi}(xv, yv, z) = \phi(xv, y, zv) = \phi(xv, yv, zv) = \phi(x, y, z)\frac{s(yz)}{s(y)s(z)},$$

$$\bar{\mathcal{R}}(x, y) = \mathcal{R}(x, y), \quad \bar{\mathcal{R}}(xv, y) = s(y)\mathcal{R}(x, y), \quad \bar{\mathcal{R}}(x, yv) = \frac{\mathcal{R}(x, y)}{s(x)},$$

$$\bar{\mathcal{R}}(xv, yv) = \mathcal{R}(x, y)\frac{s(y)}{s(x)}.$$

Proof. This is an elementary computation from the definitions of $\phi, \mathcal{R}$ for $k_F G$ and $k_{\bar{F}}\bar{G}$ and the form of $\bar{F}$ above. For example,

$$\bar{\phi}(xv, y, z) = \frac{\bar{F}(xv, y)\bar{F}(xvy, z)}{\bar{F}(y, z)\bar{F}(xv, yz)} = \frac{s(y)F(x, y)s(z)F(xy, z)}{F(y, z)s(yz)F(x, yz)} = \phi(x, y, z)\frac{s(y)s(z)}{s(yz)}$$

as stated. Similarly, $\bar{\mathcal{R}}(xv, yv) = \bar{F}(xv, yv)/\bar{F}(yv, xv) = s(y)F(x, y)/s(x)F(y, x) = \mathcal{R}(x, y)s(y)/s(x)$, etc. □

The merit of our approach is that these computations of the associator and braiding are elementary but the properties of $k_{\bar{F}}\bar{G}$ can be read off in terms of them. Thus, it is immediate that,

Corollary 3.3. *If s defines an involutive automorphism σ then $\bar{\phi} = 1$ iff $\phi = 1$, i.e. $k_{\bar{F}}\bar{G}$ is associative iff $k_F G$ is.*

Proof. In this case $\bar{\phi}$ and ϕ are given by the same expressions independent of the placement of v. □

Similarly, we gave in [1] conditions for $k_F G$ to be alternative in terms of $\mathcal{R}, \phi$. Using these, we have

Corollary 3.4. *If s defines an involutive automorphism σ then $k_{\bar{F}}\bar{G}$ is alternative iff*
(i) *$k_F G$ is alternative,*
(ii) *for all $x, y, z \in G$, either $\phi(x, y, z) = 1$ or $s(x) = s(y) = s(z) = 1$.*

Proof. Since $\bar{\phi}, \bar{\mathcal{R}}$ restrict to $\phi, \mathcal{R}$ it is immediate that $k_{\bar{F}}\bar{G}$ alternative implies $k_F G$ alternative. Here, alternativity of $k_F G$ is explicitly the condition

$$\phi(x, y, z) + \mathcal{R}(z, y)\phi(x, z, y) = 1 + \mathcal{R}(z, y),$$

$$\phi^{-1}(x, y, z) + \mathcal{R}(y, x)\phi^{-1}(y, x, z) = 1 + \mathcal{R}(y, x)$$

while for $k_{\bar{F}}\bar{G}$ we have these and other cases such as

$$\bar{\phi}(x, y, zv) + \bar{\mathcal{R}}(zv, y)\bar{\phi}(x, zv, y) = 1 + \bar{\mathcal{R}}(zv, y)$$

or, from the above results,

$$\phi(x, y, z) + s(y)\mathcal{R}(z, y)\phi(x, z, y) = 1 + s(y)\mathcal{R}(z, y).$$

Comparing, we see that in this case $(\phi(x, z, y) - 1)(s(y) - 1) = 0$. Similarly, the content of the other cases of the condition for $k_{\bar{F}}\bar{G}$ alternative is precisely $(\phi(x, z, y) - 1)(s(x) - 1) = 0$ and $(\phi(x, z, y) - 1)(s(z) - 1) = 0$ as well. Thus, $k_{\bar{F}}\bar{G}$ is alternative iff $k_F G$ is *and* for all x, y, z,

$$\phi(x, y, z) = 1, \quad \text{or} \quad s(x) = s(y) = s(z) = 1.$$

What this means is that either $k_F G$ is associative for the outcome of the corollary to hold or, if not, then σ has to be nontrivial for some of the elements whose product fails to associate. □

We now look at the case where F, s, q are of the form

$$F(x,y) = (-1)^{f(x,y)}, \quad s(x) = (-1)^{\xi(x)}, \quad q = (-1)^{\varepsilon} \tag{8}$$

for some $\mathbb{Z}_2$-valued functions f, ξ and $\varepsilon \in \mathbb{Z}_2$. We also suppose that $G = \mathbb{Z}_2^n$ and use a vector notation.

Lemma 3.5. *For G, F, s based on $\mathbb{Z}_2$, the generalised Clifford process yields the same form with $G = \mathbb{Z}_2^{n+1}$ and*

$$\bar{f}((x, x_{n+1}), (y, y_{n+1})) = f(x,y) + (y_{n+1}\varepsilon + \xi(y))x_{n+1}, \quad \bar{\xi}(x, x_{n+1}) = \xi(x) + x_{n+1}.$$

Proof. Clearly, $\bar{f}(xv, yv) = \varepsilon + \xi(y) + f(x,y)$ is the case where $x_{n+1} = y_{n+1} = 1$, while $\bar{f}(xv, y) = \xi(y) + f(x,y)$ is the case, where $x_{n+1} = 1$ and $y_{n+1} = 0$. The other two cases require to yield $f(x,y)$. The four cases can then be expressed together as stated using the field $\mathbb{Z}_2$. Similarly, for $\bar{\xi}$. □

Corollary 3.6. *Starting with k and iterating the Clifford process with a choice of $q_i = (-1)^{\varepsilon_i}$ at each step, we arrive at the standard $C(V, \mathbf{q})$ in Proposition* 2.1 *and the standard automorphism $\sigma(e_x) = (-1)^{\rho(x)} e_x$.*

Proof. We start with $f = 0$ and $\xi = 0$. Clearly, $\xi(x) = \rho(x) \bmod 2$ after n steps independent of ε_i. We also build up the expression $\sum_{j<i} x_i y_j$ in f as required, and an additional contribution to f which gives the product the expression in Proposition 2.1. □

Equivalently, we can read Lemma 3.5 inductively. If we use the notation $C(r,s)$ for the number of $\pm$ in the quadratic form then,

Corollary 3.7. *Starting with $C(r,s)$ the Clifford process with $q = 1$ yields $C(r+1, s)$. With $q = -1$ it gives $C(r, s+1)$. Hence any $C(m,n)$ with $m \geqslant r, n \geqslant s$ can be obtained by successive applications of the Clifford process from $C(r,s)$.*

Proof. $\bar{f}$ in the lemma above, given that $\xi(x) = \rho(x) \bmod 2$, is manifestly in the required form. Here, $\xi(y)x_{n+1} = \sum_{j<n+1} x_{n+1} y_j$ and $\varepsilon y_{n+1} x_{n+1}$ gives the extra factor in the product in the expression in Proposition 2.1. □

Note also that the definition of $\bar{A}$ can be written equally well as some kind of 'tensor product' $\bar{A} = A \otimes_\sigma C(k,q)$, where $C(k,q) = k[v]$ with the relation $v^2 = q$, and $\otimes_\sigma$ denotes that $\bar{A}$ factorises into these subalgebras with the cross relations $va = \sigma(a)v$ for all $a \in A$. As $k_F G$ algebras we do not need to assume an involutive automorphism σ and clearly have $k_{\bar{F}}\bar{G} = k_F G \otimes_s C(k,q)$, in general, with cross relations $v \cdot x = s(x) x \cdot v$. On the other hand, when σ is an involutive automorphism this is clearly a super tensor product of $\mathbb{Z}_2$-graded algebras.

Lemma 3.8. *When σ is an involutive automorphism as in the Clifford process, we have $\bar{A}=A\underline{\otimes}C(k,q)$ a super tensor product. Moreover, applying twice with q_1,q_2 gives*

$$\bar{\bar{A}}\cong A\underline{\otimes}C(k\oplus k,(q_1,q_2)).$$

Proof. The super tensor product $A\underline{\otimes}C(k,q_1)$ contains each factor as subalgebras with cross relations $av=(-1)^{\xi(a)}va=\sigma(a)v$, where $\xi(a)$ is the $\mathbb{Z}_2$-degree corresponding to σ. This is obviously the content of the Clifford process. Applying twice we have

$$\bar{\bar{A}}=\bar{A}\underline{\otimes}C(k,q_2)=(A\underline{\otimes}C(k,q_1))\underline{\otimes}C(k,q_2)=A\underline{\otimes}(C(k,q_1)\underline{\otimes}C(k,q_2))$$

using that the super tensor product $\underline{\otimes}$ is an associative operation. We then use Corollary 2.6. □

This super algebra periodicity can then be expressed in more usual form using the following proposition.

Proposition 3.9. *Let* $\dim(V)=2m$ *be even and σ an involutive automorphism on k_FG defined by s of the form $s(x)=(-1)^{\rho(x)}$ for a $\mathbb{Z}$-valued function ρ. Then*

$$k_FG\underline{\otimes}C(V,\mathbf{q})\cong k_{F'}G\otimes C(V,\mathbf{q}),$$

where

$$F'(x,y)=F(x,y)((-1)^{m(2m-1)}q_1\cdots q_{2m})^{(1/2)(\rho(xy)-\rho(x)-\rho(y))}.$$

If $\imath=\sqrt{-1}\in k$ then F' is cohomologous to F.

Proof. Here ρ must differ from an additive character by an even integer, hence F' is well defined. We assume that $q_i=\pm 1$ so that $\mu\equiv(-1)^{m(2m-1)}q_1\cdots q_{2m}=\pm 1$. If $\mu=1$ then $F'=F$. Otherwise, if $\imath=\sqrt{-1}\in k$ then clearly $F'=F\partial s$, where $s(x)=\imath^{-\rho(x)}$. Here, $\partial s(x,y)=s(x)s(y)/s(xy)$ is (an exact) cocycle. The same calculation shows that $(-1)^{(1/2)(\rho(xy)-\rho(x)-\rho(y))}$ is a cocycle even if $\imath\notin k$, so that F' is necessarily a cocycle. Now let $\phi:k_FG\otimes C(V,\mathbf{q})\to k_{F'}G\otimes C(V,\mathbf{q})$ be defined by $\phi(x)=x(e_1\cdots e_{2m})^{\rho(x)}$ when restricted to k_FG and the identity on $C(V,\mathbf{q})$. Here, $\gamma\equiv e_1\cdots e_{2m}$ implements the $\mathbb{Z}_2$-grading automorphism of $C(V,\mathbf{q})$ as in Corollary 2.5 and has square $\mu=\pm 1$ (these are in fact the only properties of $C(V,\mathbf{q})$ that we use, i.e. the same result applies for any super algebra with grading implemented by an element γ with $\gamma^2=\pm 1$). Then for all $x,y\in G$ we have

$$\phi(x\cdot_F y)=F(x,y)\phi(xy)=F(x,y)xy\gamma^{\rho(xy)}$$
$$=\frac{F(x,y)}{F'(x,y)}x\cdot_{F'}y\gamma^{\rho(x)+\rho(y)}\mu^{(1/2)(\rho(xy)-\rho(x)-\rho(y))}=x\gamma^{\rho(x)}y\gamma^{\rho(y)}$$

which is $\phi(x)\phi(y)$ as required. We also have

$$\phi(e_ix)=\phi(xe_i(-1)^{\rho(x)})=x\gamma^{\rho(x)}e_i(-1)^{\rho(x)}=e_ix\gamma^{\rho(x)}=\phi(e_i)\phi(x)$$

as required. Hence ϕ (which is clearly a linear isomorphism) is an algebra isomorphism as well. □

The group and cocycle F above are quite general but when we put in the form in Section 2 for the Clifford algebras we immediately obtain

$$C(V,\mathbf{q})\underline{\otimes}C(\pm)\cong C(V,-\mathbf{q})\otimes C(\pm). \tag{9}$$

Here, $C(k\oplus k,(q,q))$ is $C(2,0)=C(+)$ or $C(0,2)=C(-)$ and $F'(x,y)=F(x,y)(-1)^{x\cdot y}$ in view of (3), which changes $\mathbf{q}$ to $-\mathbf{q}$. This along with the periodicity Lemma 3.8 implies the usual periodicity properties for Clifford algebras such as

$$C(0,n+2)\cong C(n,0)\otimes\mathbb{H},\quad C(n+2,0)\cong C(0,n)\otimes M_2(k). \tag{10}$$

The additional observation $\mathbb{H}\otimes\mathbb{H}\cong M_4(k)$ then gives the usual table with period 8 for all positive or all negative signatures (constructed together). If k is algebraically closed (or at least has $\sqrt{-1}$) the situation is even simpler; we have periodicity 2 in the Clifford algebras i.e.

$$C(2m)\cong M_{2^m}(k),\quad C(2m+1)\cong M_{2^m}(k)\oplus M_{2^m}(k). \tag{11}$$

Finally, it is possible to extend the Clifford process also to representations.

Proposition 3.10. *If W is an irreducible representation of A not isomorphic to W_σ defined by the action of $\sigma(a)$ then $\bar W=W\oplus W_\sigma$ is an irreducible representation π of $\bar A$ obtained via the Clifford process with q. Here*

$$\pi(v)=\begin{pmatrix}0&1\\q&0\end{pmatrix},\quad \pi(a)=\begin{pmatrix}a&0\\0&\sigma(a)\end{pmatrix}$$

are the action on the underlying vector space $W\oplus W$ (so $\pi(a)$ is the explicit action of a in the direct sum representation $W\oplus W_\sigma$). If W,W_σ are isomorphic then W itself is an irreducible representation of $\bar A$ for a suitable value of q.

Proof. Here W_σ is the same vector space as W but with a acting by $\sigma(a)$. Clearly, we have a representation of $\bar A=A\underline{\otimes}k[v]$ since $\pi(v)\pi(a)=\pi(\sigma(a))\pi(v)$ and $\pi(v)^2=q$. If $U\subset W\oplus W$ is a nonzero subrepresentation then it is also a subrepresentation under A. The projection to the first or second part of the direct sum is A-equivariant hence its image is either 0 or W since W,W_σ are irreducible. Hence a nonzero U has dimension at least that of W. If equal in dimension then one or other projection is an isomorphism of U with W or W_σ but not both since these are not isomorphic. But in this case the form of $\pi(v)$ implies a contradiction. If greater dimension then consider the two maps $W\to W\oplus W/U$ by embedding to each summand and quotienting. The image has smaller dimension than W hence both maps are zero by W irreducible. Hence $U=W\oplus W$.

If the two representations W and W_σ are equivalent then there exists an invertible linear map $\phi:W\to W$ such that $\phi\pi(a)=\pi(\sigma(a))\phi$ where π is the action on W. We let $\pi(v)=\phi$. Note that ϕ^2 is central since σ has order 2, hence $\phi^2=q$ for some $q\in k^*$. Thus, W itself extends to an irreducible representation of $\bar A$. $\square$

Thus, k is an irreducible representation of $k=C(0,0)$ equivalent to its conjugate. Hence k is also an irreducible representation of $k[v]=C(1,0)$, the sign representation (say). This is not equivalent to its conjugate under σ (which is the trivial representation). Hence k^2 is an irreducible representation of $M_2(k)=C(2,0)$, its usual one, and so on. In this way the natural representations over any field may be mapped out.

4. Spinor representations

In this section, we use the k_FG method to obtain a new approach to the spinor representations for Clifford algebras. We have already seen that Clifford algebras may be constructed as braided-commutative algebras in a symmetric monoidal category, where the braiding has the form (5) in terms of a $\mathbb{Z}_2^n$-grading. One may make several categorical constructions along the lines of usual vector space constructions but with the braiding. For example, there is a braided tensor product algebra [9] $A\underline{\otimes}_\Psi A$ which acts on A from the left and right. The right action can be viewed as a left action using the braided-commutativity of A. Here, the braided tensor product and the action are

$$(a\otimes b)(a'\otimes b') = a\Psi(b\otimes a')b'$$
$$= aa'\otimes bb'(-1)^{\rho(b)\cdot\rho(a')+|b|\cdot|a'|}, \quad (a\otimes b).c = abc, \tag{12}$$

when A has braiding Ψ of the form in (5) (so $|\ |$ is the $\mathbb{Z}_2^n$-grading and ρ the induced $\mathbb{Z}_2$-grading). The following is a variant of this observation in which we work with the super tensor product algebra and modify the action to compensate for this.

Proposition 4.1. *Suppose that $\imath=\sqrt{-1}\in k$. If A is a $\mathbb{Z}_2^n$-graded braided-commutative algebra with respect to Ψ of the form in* (5) *then the super tensor product $A\underline{\otimes}A$ acts on A by*

$$(a\otimes b).c = abc(-1)^{|b|\cdot|c|}\imath^{\rho(b)}$$

where the $\rho(b)\equiv\sum_{i=1}^n |b|_i = |b|\cdot|b|$ is viewed in $\mathbb{Z}$ rather than in $\mathbb{Z}_2$. Moreover, the action is a $\mathbb{Z}_2^n$-graded and $\mathbb{Z}_2$-graded one.

Proof. Applying the action twice gives

$$(a\otimes b).((a'\otimes b').c) = aba'b'c(-1)^{|b'|\cdot|c|+|b|\cdot(|a'|+|b'|+|c|)}\imath^{\rho(b')+\rho(b)}$$

while the action of the super tensor product is

$$(-1)^{\rho(b)\rho(a')}(aa'\otimes bb').c = aa'bb'c(-1)^{\rho(b)\rho(a')}(-1)^{(|b|+|b'|)\cdot|c|}\imath^{\rho(b+b')}$$

which gives the same when we use braided-commutativity to write $a'b = ba'(-1)^{\rho(a')\rho(b)+|a'|\cdot|b|}$ and when we note that

$$\imath^{\rho(b+b')} = \imath^{\rho(b)}\imath^{\rho(b')}(-1)^{|b|\cdot|b'|}$$

by writing $\rho(b)=|b|\cdot|b|$ (in other words the existence of $\sqrt{-1}$ allows us to write the cocycle $(-1)^{|b|\cdot|b'|}$ as a group coboundary as in Proposition 3.9). Also, since the action is given by the product in A and this is $\mathbb{Z}_2^n$-graded it follows that the representation here is a $\mathbb{Z}_2^n$-graded and hence $\mathbb{Z}_2$-graded one as well. □

In particular, we can apply this result to any Clifford algebra acting on itself. The super tensor product algebra is a Clifford algebra on a vector space of twice the dimension by Corollary 2.6.

Corollary 4.2. *If* $\imath=\sqrt{-1}\in k$ *then* $C(V\oplus V,\mathbf{q}\oplus\mathbf{q})\cong C(V,\mathbf{q})\underline{\otimes}C(V,\mathbf{q})$ *acts on* $C(V,\mathbf{q})$ *by*

$$(e_x\otimes e_y).e_z=e_x\cdot e_y\cdot e_z(-1)^{y\cdot z}\imath^{\rho(y)}=e_{x+y+z}F(x,y)F(x+y,z)(-1)^{y\cdot z}\imath^{\rho(y)},$$

where $\rho(y)=y\cdot y\in\mathbb{Z}$. *Moreover, the action is irreducible and yields an isomorphism*

$$C(V\oplus V,\mathbf{q}\oplus\mathbf{q})\cong\mathrm{End}(C(V,\mathbf{q})).$$

Proof. Here, $C(V,\mathbf{q})$ is a super algebra with the required braided-commutativity from Corollary 2.3 as required. We write in the explicit form of the product in the basis $\{e_x\}$ and the additive group structure of $\mathbb{Z}_2^n$. This holds in fact for any $k_F\mathbb{Z}_2^n$ algebra with $\mathcal{R}$ of the required form. (Putting in F from Proposition 2.1 would give the action in the Clifford algebra case even more explicitly.) For the irreducibility and the identification with endomorphisms it suffices to show that the action is faithful (since the dimensions match). Thus, suppose that

$$0=\sum_{x,y\in\mathbb{Z}_2^n}c_{x,y}(e_x\otimes e_y).e_z=\sum_{x,y}c_{x,y}F(x,y)F(x+y,z)(-1)^{y\cdot z}\imath^{\rho(y)}e_{x+y+z}$$

for all $z\in\mathbb{Z}_2^n$. Changing variables to $x+y=x'$, as x' varies the vectors $e_{x'+z}$ run through a basis (since $\mathbb{Z}_2^n$ is a group). So

$$0=\sum_y c_{x'+y,y}F(x'+y,y)(-1)^{y\cdot z}\imath^{\rho(y)},\quad \forall z,x'.$$

We dropped the $F(x',z)$ factor here since it is nonzero for all z,x'. For each x' fixed this is the $\mathbb{Z}_2^n$ Fourier-transform of a function of y, hence the function vanishes for all y. Hence $c_{x,y}$ vanish, i.e. our action is faithful.

Also, the $\mathbb{Z}_2$-grading on the representation here is that of $C(V,\mathbf{q})$ and coincides with the canonical one induced by the action of the 'top' element $e_{(1,\dots,1)}\otimes e_{(1,\dots,1)}$ of $C(V\oplus V,\mathbf{q}\oplus\mathbf{q})$. From the above it is given by

$$\begin{aligned}(e_{(1,\dots,1)}\otimes e_{(1,\dots,1)}).e_z&=e_zF((1,\dots,1),(1,\dots,1))(-1)^{\rho(z)}\imath^n\\&=\lambda(-1)^{\rho(z)}e_z,\quad \lambda=\imath^n(-1)^{n(n-1)/2}\prod_i q_i.\end{aligned}$$ □

Our action is in closed form, but the explicit action of the generators on using the specific form of F is

$$(e_i \otimes 1).e_x = (-1)^{\sum_{j=1}^{i-1} x_j} q_i^{x_i} e_{x+(0,\dots,1,\dots,0)},$$
$$(1 \otimes e_i).e_x = \imath(-1)^{\sum_{j=1}^{i-1} x_j} (-q_i)^{x_i} e_{x+(0,\dots,1,\dots,0)}, \tag{13}$$

where $(0,\dots,1,\dots,0)$ denotes 1 in the ith place. This construction is very different from, but yields the same result as the usual construction of spinors [5] on the exterior algebra ΛV, which has the same dimension as $C(V,\mathbf{q})$. Thus, we identify the bases $\{e_{i_1}\wedge\cdots\wedge e_{i_p}\}$ and $\{e_{i_1}\cdots e_{i_p}\}$ in the two cases. With a standard choice of polarisation (or complex structure) on $V\oplus V$ the usual spinor representation (with $e_i \equiv e_i \otimes 1$ and $e_{n+i} \equiv 1 \otimes e_i$) is

$$e_i.\omega = e_i \wedge \omega + i_{e_i}\omega, \quad e_{n+i}.\omega = \imath(e_i \wedge \omega - i_{e_i}\omega) \quad \forall \omega \in \Lambda V. \tag{14}$$

Here i_v denotes the interior product in ΛV with the $\mathbf{q}$ norm. This coincides with our action on any e_x since, if $x_i = 0$ only $e_i\wedge$ contributes while if $x_i = 1$ only i_{e_i} contributes. The effect is therefore to change x_i to $x_i + 1 \bmod 2$ as in (13). The coefficients also coincide.

Next, given any super representation of a super algebra A one has a usual representation of the cross product algebra $A \rtimes k\mathbb{Z}_2$ which is the bosonisation [10] of A. The latter is just defined by adjoining a generator v with $v^2 = 1$ and cross relations $vav^{-1} = \sigma(a)$. An irreducible super representation W of A extends to the bosonistion with $v.w = \sigma_W(w) \equiv (-1)^{\rho(w)}w$ for all $w \in W$, where ρ is the super degree.

Corollary 4.3. *When* $\sqrt{-1} \in k$ *the odd Clifford algebras* $C(V \oplus V \oplus k, (\mathbf{q} \oplus \mathbf{q}, q))$ *are also represented irreducibly in the vector space of* $C(V,\mathbf{q})$. *Here, the action of* $C(V \oplus V, \mathbf{q} \oplus \mathbf{q})$ *above is extended by the additional generator* e_{2n+1} *acting as* $\lambda\sigma_W$ *with* $\lambda^2 = q$.

Proof. Here, $W = C(V,\mathbf{q})$ and $A = C(V\oplus V, \mathbf{q}\oplus\mathbf{q})$. The bosonisation consists in adjoining v which is clearly equivalent to the Clifford process. By a minor rescaling of the generator v we adopt instead the relation $v^2 = q$ for the Clifford process with parameter q and identify it with e_{2n+1} of $C(V\oplus V\oplus k, (\mathbf{q}\oplus\mathbf{q}, q))$. Note that W remains irreducible since any submodule restricted to $C(V \oplus V, \mathbf{q} \oplus \mathbf{q})$ must coincide with W. $\square$

One can also endow $A \rtimes k\mathbb{Z}_2$ with a new super-algebra structure, with v of degree 1. The extended representation W is no longer a super representation but one can be obtained by doubling it to $\bar{W} = W \oplus W$. Thus, $C(V,\mathbf{q}) \oplus C(V,\mathbf{q}) \cong C(V \oplus k, (\mathbf{q}, q))$ becomes a super representation of the odd Clifford algebra. In this case, applying the bosonisation again gives a representation equivalent to the next higher even Clifford algebra representation acting as in the above Corollary but acting on $C(V \oplus k, (\mathbf{q}, q))$. One can also view these results from the Clifford process point of view in the previous section.

As a very concrete example of our main result, consider the spinor representation for the Clifford algebra $C(0,4)$ in 4 Euclidean dimensions. We work over $\mathbb{C}$ and by the above this can be considered as $\mathbb{H}\underline{\otimes}\mathbb{H}$ acting on $\mathbb{H}$, where $\mathbb{H}=C(0,2)$ is the complex quaternions. Thus, a 'Dirac spinor' in physics is nothing other than an $\mathbb{H}$-valued function. With basis $\{e_1,e_2\}$ of the 2-dimensional vector space V taken as the generators of $\mathbb{H}$, the spinor action from Corollary 4.2 is

$$(e_i\otimes 1).\psi=e_i\psi,\quad (1\otimes e_i).\psi=\imath e_i\psi(-1)^{|\psi|_i} \tag{15}$$

on a spinor of homogeneous degree $|\psi|\in\mathbb{Z}_2^2$. The right-hand side here uses the quaternion product. The construction of spinor representations in terms of left and right actions of quaternions has previously been alluded to in some contexts in the literature, see for example [3]. However, we are not aware of a general treatment as above.

As an application, the standard Dirac operator on the 4-dimensional space $V\oplus V$ under the identification of Corollary 2.6 is

$$\partial\!\!\!/\psi=(e_1\otimes 1)\partial_1+(e_2\otimes 1)\partial_2+(1\otimes e_1)\partial_3+(1\otimes e_2)\partial_4$$

where ∂_1 is the differentiation in the first basis direction of $V\oplus V$, etc. So this becomes

$$\partial\!\!\!/\psi=e_1(\partial_1+\imath\partial_3(-1)^{|\psi|_1})\psi+e_2(\partial_2+\imath\partial_4(-1)^{|\psi|_2})\psi. \tag{16}$$

It is possible to make this more explicit by taking a basis of $\mathbb{H}$, namely $1,e_1,e_2$ and $e_3\equiv e_1e_2$. Then

$$e_ie_j=-\delta_{ij}+\varepsilon_{ijk}e_k,\quad i,j,k=1,2,3$$

as usual, in terms of the Kronecker delta-function and the totally antisymmetric tensor with $\varepsilon_{123}=1$. We write a spinor as an ordered pair $\psi=(\psi_0,\psi_i)$ with $i=1,2,3$ according to the components in this basis. Finally, we write

$$\nabla_1\equiv\partial_1+\imath\partial_3,\quad \nabla_2\equiv\partial_2+\imath\partial_4$$

and denote by $\bar\nabla$ the same expressions with $-\imath$. Then

$$(\partial\!\!\!/\psi)_0=-\bar\nabla_1\psi_1-\bar\nabla_2\psi_2,\quad (\partial\!\!\!/\psi)_1=\nabla_1\psi_0+\bar\nabla_2\psi_3,$$

$$(\partial\!\!\!/\psi)_2=\nabla_2\psi_0-\bar\nabla_1\psi_3,\quad (\partial\!\!\!/\psi)_3=\nabla_1\psi_2-\nabla_2\psi_1$$

using the relations in $\mathbb{H}$. If we define $\nabla_3\equiv 0$ and $\vec\psi=(\psi_1,\psi_2,\bar\psi_3)$ then this can be written compactly as

$$(\partial\!\!\!/\psi)_0=-\bar\nabla\cdot\vec\psi,\quad \vec{\partial\!\!\!/}\psi=\nabla\psi_0+\bar\nabla\times\bar{\vec\psi} \tag{17}$$

in terms of usual divergence, gradient and curl in 3 (complex) dimensions and pointwise complex conjugation.

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